

# $G_\delta$ -REFINEMENTS

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**ABSTRACT.** In this work we deal with the preservation by  $G_\delta$ -refinements. We prove that for SP-scattered spaces the metacompactness, paralindelöfness, metalindelöfness and linear lindelöfness are preserved by  $G_\delta$ -refinements. In this context we also consider some other generalizations of discrete spaces like  $\omega$ -scattered and  $N$ -scattered. In the final part of this paper we look at a question of Juhász, Soukup, Szentmiklóssy and Weiss concerning the tightness of the  $G_\delta$ -refinement of a  $\sigma$ -product.

## 1. PRELIMINARIES

**Definition 1.1.** For any space  $\langle X, \tau \rangle$ , the topology  $\tau_\delta$  obtained by letting every  $G_\delta$  subset of  $X$  be open is called the  $G_\delta$ -**topology** and the space so obtained is denoted by  $X_\delta$ .

**Definition 1.2.** Let  $X$  be a set and let  $\mu$  be a cardinal such that  $\mu \leq |X|$ . We say that  $\mathcal{C} \subseteq [X]^\mu$  is *cofinal* in  $[X]^\mu$  if, for all  $x \in [X]^\mu$ , there exists  $y \in \mathcal{C}$  such that  $x \subseteq y$ . For cardinals  $\mu \leq \kappa$ , we define  $\text{cf}([ \kappa ]^\mu, \subseteq)$  as the least cardinality of a cofinal family in  $[ \kappa ]^\mu$ . Given an infinite cardinal  $\kappa$ , we define  $\text{Cov}_\omega(\kappa) = \text{cf}([ \kappa ]^{\aleph_0}, \subseteq)$ .

**Theorem 1.3** (Passos [2]). *Let  $\kappa$  be an infinite cardinal such that  $\text{Cov}_\omega(\kappa) = \kappa$ . Given a set  $X$  of cardinality  $\kappa$ , there exists an  $\omega$ -covering elementary submodel  $M$  such that  $X \subseteq M$  and  $|M| = \kappa$ .*

Recall that a subset  $F$  of  $X$  is  $\kappa$ -closed, where  $\kappa$  is an infinite cardinal iff whenever  $S \subseteq F$  and  $|S| \leq \kappa$  then  $S \subseteq F$ . It is well known that  $t(X) \leq \kappa$  iff every  $\kappa$ -closed set in  $X$  is closed.

## 2. SP-SCATTERED SPACES

**Definition 2.1.** A point  $p$  in a topological space  $X$  is called a *strong  $P$ -point* if it has a neighborhood consisting of  $P$ -points. The set of all strong  $P$ -points of  $X$  is denoted by  $\text{SP}(X)$ .

Observe that  $\text{SP}(X) = \text{int}_X P(X)$ .

**Definition 2.2** ([5]). Recursively, define:

- $S_0(X) = X$  and  $S_1(X) = X \setminus \text{SP}(X)$ ;
- $S_{\alpha+1}(X) = S_1(S_\alpha(X))$  for any ordinal  $\alpha \geq 1$ ;
- $S_\lambda(X) = \bigcap \{ S_\alpha(X) : \alpha < \lambda \}$ , if  $\lambda$  is a limit ordinal.

In [5], Henriksen, Raphael and Woods proved the following generalizations of the well known theorems 5.1 and 5.2 of [3], respectively:

**Theorem 2.3.** *If  $X$  is a Lindelöf SP-scattered regular space, then  $X_\delta$  is Lindelöf.*

**Theorem 2.4.** *If  $X$  is a paracompact SP-scattered Hausdorff space, then  $X_\delta$  is paracompact.*

In the same article they asked:

**Question 2.5.** *If  $X$  is a metacompact SP-scattered regular space, so is  $X_\delta$  a metacompact space?*

In this section, we will see that not just the metacompactness, but also the paralindelöfness, the metalindelöfness and the linear lindelöfness are preserved by  $G_\delta$ -refinements on the class of SP-scattered regular spaces.

**Theorem 2.6** ([4]). *A regular  $P$ -space  $X$  is paralindelöf if, and only if, it is paracompact.*

**Proposition 2.7** ([5]). *If  $X$  is a regular space, then the following are equivalent:*

- (1)  $X$  is SP-scattered;
- (2) if  $A \subseteq X$  is nonempty, then  $\text{int}_A \{ a : a \text{ is a } P\text{-point of } A \} \neq \emptyset$ .

**Theorem 2.8.** *If  $X$  is a regular SP-scattered paralindelöf space, then  $X_\delta$  is paracompact.*

*Proof.* By the theorem 2.6, it is enough to show that  $X_\delta$  is paralindelöf. Let  $\mathcal{C}$  be an open cover of  $X_\delta$ . Let  $O$  be the set of all points  $x \in X$  such that  $x \in \text{int}_\tau \bigcup \mathcal{C}'$  for some locally countable open partial refinement  $\mathcal{C}'$  of  $\mathcal{C}$  in  $X_\delta$ .

If  $O = X$  then, for each  $x \in X$ , there exists a locally countable open partial refinement  $\mathcal{C}_x$  of  $\mathcal{C}$  in  $X_\delta$  such that  $x \in V_x = \text{int}_\tau \bigcup \mathcal{C}_x$ . Since  $\langle X, \tau \rangle$  is paralindelöf, the open cover  $\mathcal{V} = \{ V_x : x \in X \}$  admits a locally

countable open refinement  $\{W_s : s \in S\}$ , with  $W_s \neq W_{s'}$  whenever  $s \neq s'$ . For each  $s \in S$ , take  $x_s \in X$  such that  $W_s \subseteq V_{x_s}$ . So

$$\{W_s \cap C : s \in S \text{ and } C \in \mathcal{C}_{x_s}\}$$

is locally countable open refinement of  $\mathcal{C}$  in  $X_\delta$ .

Now it remains to show that in fact  $O = X$ . Suppose this is not the case. Since  $\langle X, \tau \rangle$  is SP-scattered, by the proposition 2.7, there exists a  $y \in X \setminus O$  and an open neighborhood  $U$  of  $y$  in  $X$  such that  $(X \setminus O) \cap U$  is a  $P$ -subspace of  $X$ . Take  $C_y \in \mathcal{C}$  such that  $y \in C_y$ . We can suppose that  $C_y = \bigcap \{U_n : n \in \omega\}$ , where, for each  $n \in \omega$ ,  $U_n \in \tau$  and  $\text{cl}_\tau(U_n) \subseteq U$ . So  $C_y \cup O$  is an open subset of  $X$ , for  $(X \setminus O) \cap C_y$  is a subspace of the  $P$ -space  $U$ . As  $X$  is regular,  $y$  has an open neighborhood  $U_y$  in  $X$  such that  $\text{cl}_\tau(U_y) \subseteq C_y \cup O$ .

Fix  $n \in \omega$ . Let  $F = \text{cl}_\tau(U_y) \setminus U_n$ . Since  $F \subseteq O$ , for each  $x \in F$ , there exists a locally countable open partial refinement  $\mathcal{C}_x$  of  $\mathcal{C}$  in  $X_\delta$  such that  $x \in V_x = \text{int}_\tau \bigcup \mathcal{C}_x$ . As  $\langle X, \tau \rangle$  is paralindelöf and  $F$  is closed,  $\mathcal{V} = \{V_x : x \in F\}$  admits a locally countable open partial refinement  $\mathcal{W} = \{W_s : s \in S\}$  that covers  $F$ , where  $W_s \neq W_{s'}$  whenever  $s \neq s'$ . For each  $s \in S$ , choose  $x_s \in F$  such that  $W_s \subseteq V_{x_s}$ . Consider the family

$$\mathcal{D}_n = \{W_s \cap C : s \in S \text{ e } C \in \mathcal{C}_{x_s}\}.$$

The family  $\mathcal{D}_n$  is a locally countable open cover of  $F$  in  $X_\delta$ . Indeed, let  $x \in X$ . Since  $\mathcal{W}$  is a locally countable open family in  $X$ , there exists an open neighborhood  $Z_x$  of  $x$  in  $X$  such that  $T = \{s \in S : W_s \cap Z_x \neq \emptyset\}$  is countable. For each  $t \in T$ , take an open neighborhood  $O_t$  of  $x$  in  $X_\delta$  such that  $\{C \in \mathcal{C}_{x_t} : C \cap O_t \neq \emptyset\}$  is countable. Consider the following open neighborhood of  $x$  in  $X_\delta$ :

$$Z = Z_x \cap \bigcap \{O_t : t \in T\}.$$

Note that for each  $t \in T$ ,  $\mathcal{C}_{x_t}(Z) = \{C \in \mathcal{C}_{x_t} : C \cap Z \neq \emptyset\}$  is countable. Seeing that

$$\begin{aligned} \mathcal{D}_n(Z) &= \{D \in \mathcal{D}_n : D \cap Z \neq \emptyset\} \\ &= \{W_i \cap C : s \in S, C \in \mathcal{C}_{x_s} \text{ e } W_s \cap C \cap Z \neq \emptyset\} \\ &\subseteq \{W_t \cap C : t \in T \text{ e } C \in \mathcal{C}_{x_t}(Z)\}, \end{aligned}$$

the family  $\mathcal{D}_n(Z)$  is countable.

Thus,

$$\mathcal{C}' = \{C_y\} \cup \bigcup \{\mathcal{D}_n : n \in \omega\}$$

is a locally countable open partial refinement of  $\mathcal{C}$  in  $X_\delta$  such that  $y \in U_y \subseteq \text{int}_\tau \bigcup \mathcal{C}'$ , contradicting the fact that  $y \notin O$ .  $\square$

**Theorem 2.9.** *If  $X$  is a regular SP-scattered metalindelöf space, then  $X_\delta$  is metalindelöf.*

*Proof.* Let  $\mathcal{C}$  be an open cover of  $X_\delta$  and consider the set  $O$  whose elements are all those  $x \in X$  such that  $x \in \text{int}_\tau \bigcup \mathcal{C}'$  for some pointwise countable open partial refinement  $\mathcal{C}'$  of  $\mathcal{C}$  in  $X_\delta$ .

If  $O = X$  then, proceeding in the same way as in the proof of theorem 2.8, we can obtain a pointwise countable open refinement of  $\mathcal{C}$  in  $X_\delta$ .

In order to complete the proof, it is enough to show that  $O = X$ . Suppose on the contrary that  $O \neq X$ . As  $X$  is regular and SP-scattered, by the proposition 2.7, there exist a point  $y \in X \setminus O$  and an open neighborhood  $U$  of  $y$  in  $X$  such that  $U \cap (X \setminus O)$  is a  $P$ -subspace of  $X$ . Choose a  $C_y \in \mathcal{C}$  such that  $y \in C_y$ . We can suppose that  $C_y = \bigcap \{U_n : n \in \omega\}$ , where for each  $n \in \omega$ ,  $U_n \in \tau$  and  $\text{cl}_\tau(U_n) \subseteq U$ . Note that  $C_y \cup O$  is an open subset of  $X$ . Hence, from the regularity of  $X$  it follows that there exist an open neighborhood  $U_y$  of  $y$  in  $X$  such that  $\text{cl}_\tau(U_y) \subseteq C_y \cup O$ .

Fix  $n \in \omega$ . Note that  $F = \text{cl}_\tau(U_y) \setminus U_n \subseteq O$ . Then, for each  $x \in F$ , there is a pointwise countable open partial refinement  $\mathcal{C}_x$  of  $\mathcal{C}$  in  $X_\delta$  such that  $x \in V_x = \text{int}_\tau(\bigcup \mathcal{C}_x)$ . Because  $X$  is metalindelöf and  $F$  is closed then  $\{V_x : x \in F\}$  has a pointwise countable open partial refinement  $\mathcal{W} = \{W_i : i \in I\}$ , where  $W_i \neq W_j$  whenever  $i \neq j$ . For each  $i \in I$ , choose  $x_i \in F$  such that  $W_i \subseteq V_{x_i}$ . Consider the family

$$\mathcal{D}_n = \{W_i \cap C : i \in I \text{ e } C \in \mathcal{C}_{x_i}\}.$$

It is easily checked that each  $\mathcal{D}_n$  is a pointwise countable open partial refinement of  $\mathcal{C}$  which covers  $\text{cl}_\tau(U_y) \setminus U_n$ . So,

$$\mathcal{C}' = \{C_y\} \cup \bigcup \{\mathcal{D}_n : n \in \omega\}$$

is a pointwise countable open partial refinement of  $\mathcal{C}$  in  $X_\delta$  such that  $y \in U_y \subseteq \text{int}_\tau(\bigcup \mathcal{C}')$ , contradicting the fact that  $y \notin O$ . Thus,  $O = X$ .  $\square$

**Theorem 2.10.** *If  $X$  is a regular SP-scattered metacompact space, then  $X_\delta$  is metacompact.*

*Proof.* Let  $\mathcal{C}$  be an open cover of  $X_\delta$  and consider the set  $O$  whose elements are all  $x \in X$  such that  $x \in \text{int}_\tau(\bigcup \mathcal{C}')$  for some pointwise finite open partial refinement  $\mathcal{C}'$  of  $\mathcal{C}$  in  $X_\delta$ .

Similarly to what it has been done in theorem 2.8, we can get, from the assumption  $O = X$ , a pointwise finite open refinement of  $\mathcal{C}$  in  $X_\delta$ .

We complete the proof by showing that  $O = X$ . Suppose that  $O \neq X$ . Since  $\langle X, \tau \rangle$  is SP-scattered and regular, by the proposition 2.7, there are a point  $y \in X \setminus O$  and an open neighborhood  $U$  of  $y$  in  $X$  such that  $(X \setminus O) \cap U$  is a  $P$ -subspace of  $X$ . Take a  $C_y \in \mathcal{C}$  such that  $y \in C_y$ . We can suppose that  $C_y = \bigcap \{U_n : n \in \omega\}$ , where for each  $n \in \omega$ ,  $U_n \in \tau$  and  $\text{cl}_\tau(U_{n+1}) \subseteq U_n \subseteq \text{cl}_\tau(U_n) \subseteq U$ . Note that  $C_y \cup O$  is an open subset of  $X$ . Once  $X$  is regular,  $y$  has an open neighborhood  $U_y$  in  $X$  such that  $\text{cl}_\tau(U_y) \subseteq (C_y \cup O) \cap U_0$ .

As  $F_n \subseteq O$ , for each  $x \in F_n$ , there exists a pointwise finite open partial refinement  $\mathcal{C}_x$  of  $\mathcal{C}$  in  $X_\delta$  such that  $x \in V_x = \text{int}_\tau(\bigcup \mathcal{C}_x)$ . Since  $\langle X, \tau \rangle$  is metacompact and  $F_n$  is closed then  $\mathcal{V} = \{V_x : x \in F_n\}$  has a pointwise finite open partial refinement  $\mathcal{W}$  which covers  $F_n$ . We can suppose that  $\mathcal{W} = \{W_i : i \in I\}$ , where  $W_i \neq W_j$  whenever  $i \neq j$ . For each  $i \in I$ , choose  $x_i \in F_n$  such that  $W_i \subseteq V_{x_i}$ . Consider the family

$$\mathcal{D}_n = \{W_i \cap C \cap (U_n \setminus \text{cl}_\tau(U_{n+2})) : i \in I \text{ and } C \in \mathcal{C}_{x_i}\}.$$

Note that each  $\mathcal{D}_n$  is a pointwise finite open partial refinement of  $\mathcal{C}$  in  $X_\delta$  such that  $\bigcup \mathcal{D}_n = U_n \setminus \text{cl}_\tau(U_{n+2})$ . Therefore,

$$\mathcal{C}' = \{C_y\} \cup \bigcup \{\mathcal{D}_n : n \in \omega\}$$

is a pointwise finite open partial refinement of  $\mathcal{C}$  in  $X_\delta$  such that  $y \in U_y \subseteq \text{int}_\tau(\bigcup \mathcal{C}')$ , contradicting the fact that  $y \notin O$ .  $\square$

**Theorem 2.11.** *If  $X$  is a regular SP-scattered linearly Lindelöf space, then  $X_\delta$  is linearly Lindelöf.*

*Proof.* Let  $\mathcal{C} = \{C_\alpha : \alpha < \kappa\}$  be an open cover of  $X_\delta$ , where  $\kappa$  is an uncountable regular cardinal. For each  $\alpha < \kappa$ , let

$$V_\alpha = \text{int}_\tau \left( \bigcup \{C_\beta : \beta \leq \alpha\} \right).$$

Define

$$O = \{x \in X : \text{there exists } \alpha(x) < \kappa \text{ such that } x \in V_{\alpha(x)}\}.$$

**Claim.**  $O = X$

**Proof of claim.** Suppose on the contrary that  $O \neq X$ . As  $\langle X, \tau \rangle$  is a SP-scattered regular space, by the proposition 2.7, there are  $y \in X \setminus O$  and an open neighborhood  $U$  of  $y$  in  $X$  such that  $(X \setminus O) \cap U$  is a  $P$ -subspace of  $X$ . Choose  $\alpha_y < \kappa$  such that  $y \in C_{\alpha_y}$ . We can suppose that  $C_{\alpha_y} = \bigcap \{U_n : n \in \omega\}$ , where, for each  $n \in \omega$ ,  $U_n \in \tau$  and  $\text{cl}_\tau(U_n) \subseteq U$ . Note that  $C_y \cup O$  is an open subset of  $X$ . Once  $X$  is regular,  $y$  has an open neighborhood  $U_y$  in  $X$  such that  $\text{cl}_\tau(U_y) \subseteq C_{\alpha_y} \cup O$ .

Fix  $n \in \omega$ . Let  $F_n = \text{cl}_\tau(U_y) \setminus U_n$ . Note that  $F_n$  is a closed subset of  $X$  and so it is linearly Lindelöf. Moreover,  $F_n \subseteq O$ ; this implies that, for each  $x \in F_n$ , we can take  $\alpha(x) < \kappa$  such that  $x \in V_{\alpha(x)}$ . Then,  $\{V_{\alpha(x)} : x \in F_n\}$  is a family of open subsets of  $X$  which covers  $F_n$  and it is linearly ordered by inclusion. Therefore, there is a countable subset  $E_n \subseteq F_n$  such that  $\{V_{\alpha(x)} : x \in E_n\}$  covers  $F_n$ . So,  $\alpha = \sup(\{\alpha_y\} \cup \bigcup \{E_n : n \in \omega\}) < \kappa$  e

$$y \in U_y \subseteq C_{\alpha_y} \cup \bigcup \{F_n : n \in \omega\} \subseteq \bigcup \{C_\beta : \beta \leq \alpha\}.$$

Then  $y \in V_\alpha$  and, thus,  $y \in O$ . This is a contradiction.  $\triangle$

By the claim above,  $\mathcal{V} = \{V_\alpha : \alpha < \kappa\}$  is an open cover of  $\langle X, \tau \rangle$ . Since  $\langle X, \tau \rangle$  is a linearly Lindelöf space,  $\mathcal{V}$  has a subcover whose cardinality is less than  $\kappa$ . Because  $\kappa$  is regular,  $V_\alpha = X$  for some  $\alpha < \kappa$ . Thus,  $\{C_\beta : \beta < \alpha\}$  is a subcover of  $\mathcal{C}$  whose cardinality is less than  $\kappa$ .  $\square$

### 3. OTHER GENERALIZATIONS OF SCATTERED

Clearly, if a regular Lindelöf space  $X$  is a countable union of scattered closed subspaces, then  $X_\delta$  is Lindelöf. As we shall see, at least consistently, this is not the case when it is not required that the subspaces are closed.

A space is  **$\sigma$ -scattered** if it is an union of a countable family of scattered subspaces.

**Example 1.** *Assuming CH, there exists a regular  $\sigma$ -scattered Lindelöf space whose  $G_\delta$ -refinement is not Lindelöf.*

*Proof.* It is enough to take a Luzin subset of the real line containing the rational numbers and consider it as a subspace of the Michael line.  $\square$

**Question 3.1.** Is there a regular  $\sigma$ -scattered Lindelöf space whose  $G_\delta$ -refinement is not Lindelöf?

**Question 3.2.** Is there a regular  $\sigma$ -scattered paracompact space whose  $G_\delta$ -refinement is not paracompact?

Hdeib and Pareek introduced in [6] the following natural generalization of scattered spaces: a space  $X$  is  $\omega$ -**scattered** if, for each non-empty subset  $A$  of  $X$ , there exist a point  $x \in A$  and an open neighborhood  $U_x$  of  $x$  such that  $U_x \cap A$  is countable.

Every scattered space is  $\omega$ -scattered, but the reverse is not true: the set of rational numbers with the usual topology is  $\omega$ -scattered and non-scattered.

The theorem 3.12 of [6] states that in the class of regular  $\omega$ -scattered spaces the Lindelöf property is preserved by  $G_\delta$ -refinements. However, this is not true once the space of the example 1 is  $\omega$ -scattered.

**Question 3.3.** Is there a Hausdorff  $\omega$ -scattered paracompact space  $X$  such that  $X_\delta$  is not paracompact?

A space  $X$  is  $N$ -**scattered** if every nowhere dense subset of  $X$  is a scattered subspace of  $X$ . The next example was noticed by Santi Spadaro.

**Example 2.** Assuming CH, there exists a  $N$ -scattered Lindelöf space whose  $G_\delta$ -refinement is not Lindelöf.

*Proof.* Let  $\mathcal{M}$  the family of all Lebesgue measurable subsets of the real line. For each  $E \in \mathcal{M}$ , define

$$\Phi(E) = \left\{ x \in \mathbb{R} : \lim_{h \rightarrow 0} \frac{m(E \cap ]x-h, x+h])}{2h} = 1 \right\}.$$

Then

$$\tau_d = \{ E \in \mathcal{M} : E \subseteq \Phi(E) \}$$

is a topology on  $\mathbb{R}$  stronger than that usual, well known as *density topology*. Denote by  $\mathbb{R}_d$  the topological space  $\langle \mathbb{R}, \tau_d \rangle$ . By corollary 4.3 of [7], CH implies that  $\langle \mathbb{R}, \tau_d \rangle$  has a hereditarily Lindelöf, non-separable, regular and Baire subspace  $X$ . By theorem 2.7 of [7], every nowhere dense subset of  $X$  is discrete (and closed). Therefore,  $X$  is  $N$ -scattered. On the other side, the pseudocharacter of  $X$  is countable, for  $X$  is Hausdorff and hereditarily Lindelöf. Then  $X_\delta$  is discrete and uncountable and, thus, it is not Lindelöf.  $\square$

**Question 3.4.** Is there a Hausdorff paracompact  $N$ -scattered space whose  $G_\delta$ -refinement is not a paracompact space?

#### 4. THE TIGHTNESS OF $G_\delta$ -REFINEMENT OF $\sigma$ -PRODUCTS

Given a family of topological spaces  $\{X_i : i \in I\}$  and a point  $x^* \in X = \prod\{X_i : i \in I\}$ , define

$$\sigma = \sigma(X) = \sigma(X, x^*) = \left\{ x \in \prod_{i \in I} X_i : \text{supp}(x) \text{ is finite} \right\},$$

where  $\text{supp}(x) = \{i \in I : x(i) \neq x^*(i)\}$ . The  $\sigma$ -product of  $X$  at  $x^*$  is the set  $\sigma$  equipped with the topology induced by the Tychonoff product  $\prod\{X_i : i \in I\}$ .

In [1], Juhász, Soukup, Szentmiklóssy and Weiss proved:

**Theorem 4.1.** Let  $\kappa$  and  $\lambda$  be cardinals, with  $\kappa \leq \aleph_1$ . Let  $X$  be the one point lindelöfication of a discrete space of cardinality  $\kappa$  by a point  $p$  and let  $x^* \in X^\kappa$ , where  $x^*(\alpha) = p$  for all  $\alpha < \kappa$ . Then  $(\sigma(X^\kappa, x^*))_\delta$  has tightness  $\aleph_1$ .

In the same article, it was asked:

**Question 4.2.** Assume that  $X$  is a Lindelöf  $P$ -space such that  $t(X) = \aleph_1$ . Is it true that

$$t(\sigma(X^\kappa)_\delta) = \aleph_1$$

for all cardinal  $\kappa$ ?

We will see that the answer is positive.

**Lemma 4.3** ([8]). If  $\{X_i : i \leq n\}$  is a finite family of regular locally Lindelöf  $P$ -spaces, then

$$t\left(\prod\{X_i : i \leq n\}\right) = \max\{t(X_i) : i \leq n\}.$$

**Lemma 4.4.** If  $X = \sigma\{X_n : n \in \omega\}$  is a  $\sigma$ -product of regular locally Lindelöf  $P$ -spaces, then

$$t(X_\delta) = \sup\{t(X_n) : n \in \omega\}.$$

*Proof.* Let  $\lambda = \sup\{t(X_n) : n \in \omega\}$ . Let  $Y$  be a non-closed subset of  $\sigma_\delta = \sigma(X, x^*)_\delta$  and let  $q \in \text{cl}(Y) \setminus Y$ . For each  $n \in \omega$ , let

$$Y_n = \{y \in Y : \text{supp}(y) \subseteq n\}.$$

Since  $Y = \bigcup\{Y_n : n \in \omega\}$  and  $\sigma(X, x^*)_\delta$  is a  $P$ -space, there exists a  $m \in \omega$  such that  $q \in \text{cl}(Y_m)$ . Now,  $\pi_m(q) \in \text{cl}(\pi_m[Y_m])$ , where  $\pi_m$  is the natural projection from  $\prod\{X_n : n \in \omega\}$  in  $\prod\{X_i : i \in m\}$ . Since by the corollary 4.3 the tightness of  $\prod\{X_i : i \in m\}$  is  $\leq \lambda$ , there exists  $Z' \subseteq \pi_m[Y_m]$  of cardinality  $\leq \lambda$  such that  $\pi_m(q) \in \text{cl}(Z')$ . Then  $Z = Z' \times \prod\{x^*(n) : n \geq m\} \subseteq Y_m \subseteq Y$  and since  $\text{supp}(q) \subseteq m$ , we have  $q \in \text{cl}(Z)$ .  $\square$

**Lemma 4.5.** *Let  $\kappa$  be a infinite cardinal. Let  $X = \prod\{X_\alpha : \alpha < \kappa\}$ . If for each countable subset  $I \subseteq \kappa$ ,  $\sigma\{X_\alpha : \alpha \in I\}_\delta$  has tightness  $\leq \lambda$ , with  $\text{Cov}_\omega(\lambda) = \lambda$ , then  $\sigma(X)_\delta$  has tightness  $\leq \lambda$ .*

*Proof.* Let  $Y$  be a non-closed subset of  $\sigma_\delta = \sigma(X, x^*)_\delta$  and let  $q \in \text{cl}(Y) \setminus Y$ . By the theorem 1.3, there exists a  $\omega$ -covering elementary submodel  $M$  of cardinality  $\lambda$  such that  $\{X, x^*, \kappa, \lambda, q, Y\} \cup \lambda \subseteq M$ . We are going to show that  $q \in \text{cl}(Y \cap M)$ . Suppose that

$$q \in U = \prod\{U_\alpha : \alpha \in I\} \times \prod\{X_\alpha : \alpha \in \kappa \setminus I\},$$

where  $I$  is a countable subset of  $\kappa$  and each  $U_\alpha$  is an open subset of  $X_\alpha$ . Since  $M$  is  $\omega$ -covering, there exists  $J \in M$ , a countable subset of  $\kappa$  such that  $I \cap M \subseteq J$ . Now note that  $\pi_J(q) \in \text{cl}(\pi_J[Y])$  and  $\pi_J[Y] \in M$ ; besides  $\pi_J[\sigma(X, x^*)_\delta]$  belongs to  $M$  and, by hypothesis, its tightness is  $\leq \lambda$ . So by the elementarity there exists  $Z \in M$ , a subset of  $\pi_J[Y]$  whose cardinality is at most  $\lambda$ , such that  $\pi_J(q) \in \text{cl}(Z)$ . Let  $z \in \pi_J[U] \cap Z$ . Note that  $z \in M$ , because, since  $Z \in M$  and  $Z$  has cardinality at most  $\lambda$  and  $\lambda \subseteq M$ ,  $Z \subseteq M$ . So by the elementarity there exists  $y \in Y \cap M$  such that  $\pi_J(y) = z$ .

We claim that  $y \in U$ . Indeed, since  $\text{supp}(y), \text{supp}(q) \subseteq M$ , it follows that if  $\alpha \in I \setminus M$  so  $y(\alpha) = x^*(\alpha) = q(\alpha) \in U_\alpha$ . On the other side, if  $\alpha \in I \cap M$  so  $\alpha \in J$  and thus  $y(\alpha) = z(\alpha) \in U_\alpha \cap M$ . Therefore,  $y \in \prod\{U_\alpha : \alpha \in I\} \times \prod\{X_\alpha : \alpha \in \kappa \setminus I\}$ .  $\square$

**Theorem 4.6.** *If  $X = \prod\{X_\alpha : \alpha < \kappa\}$ , with each  $X_\alpha$  being a Lindelöf  $P$ -space such that  $t(X_\alpha) \leq \lambda$ , with  $\text{Cov}_\omega(\lambda) = \lambda$ , then*

$$t(\sigma(X)_\delta) \leq \lambda.$$

*In particular, if  $X$  is a Lindelöf  $P$ -space whose tightness is  $\aleph_n$  then the tightness of  $\sigma(X^\kappa, x^*)_\delta$  is  $\aleph_n$ .*

As a corollary of the previous theorem we have that, for a regular Lindelöf  $P$ -space,

$$t(\sigma(X^\kappa)_\delta) \leq t(X)^{\aleph_0}.$$

It remains to be seen whether:

**Question 4.7.** Is there a Lindelöf  $P$ -space  $X$  such that  $t(X) = \lambda$ , with  $\text{Cov}_\omega(\lambda) > \lambda$ , and

$$t(\sigma(X^\kappa)_\delta) > \lambda?$$

**Question 4.8.** Assuming that  $\text{Cov}_\omega(\aleph_\omega) = \aleph_{\omega+1}$ , is there a Lindelöf  $P$ -space  $X$  such that  $t(X) = \aleph_\omega$  and

$$t(\sigma(X^\kappa)_\delta) = \aleph_{\omega+1}?$$

Based on the theorem 3.1 from [8], we have:

**Lemma 4.9.** *Let  $\lambda$  be a infinite cardinal. If  $X = \sigma\{X_\alpha : \alpha < \lambda\}$  is a  $\sigma$ -product of regular Lindelöf  $P$ -spaces, then*

$$t(X_\delta) \leq \text{Cov}_\omega(\lambda) \cdot \sup\{t(X_\alpha) : \alpha < \lambda\}.$$

*Proof.* Let  $\kappa = \text{Cov}_\omega(\lambda) \cdot \sup\{t(X_\alpha) : \alpha < \lambda\}$ . For each  $I \subseteq \lambda$ , let  $\sigma_I = \sigma\{X_\alpha : \alpha \in I\}_\delta$ . Suppose that  $A \subseteq \sigma_\delta$  is  $\kappa$ -closed, and  $a \in \text{cl}_{\sigma_\delta}(A)$ .

Note that, for each countable subset  $J \subseteq \lambda$ ,  $\pi_J[\text{cl}_{\sigma(\lambda)}(A)] \subseteq \text{cl}_{\sigma_J}(\pi_J[A])$ . Indeed, let  $x \in \pi_J[\text{cl}_{\sigma(\lambda)}(A)]$ . Then there exists  $z \in \text{cl}_{\sigma(\lambda)}(A)$  such that  $\pi_J(z) = x$ . If  $\prod\{U_j : j \in J\}$  is an basic neighborhood of  $x$  in  $\sigma_J$ , then  $\prod\{U_j : j \in J\} \times \prod\{X_\alpha : \alpha \in \lambda \setminus J\}$  is an open neighborhood of  $z$ . So  $(\prod\{U_j : j \in J\} \times \prod\{X_\alpha : \alpha \in \lambda \setminus J\}) \cap A \neq \emptyset$  and, thus,  $(\prod\{U_j : j \in J\}) \cap \pi_J[A] \neq \emptyset$ . Therefore,  $x \in \text{cl}_{\sigma_J}(\pi_J[A])$ .

Then, for each countable subset  $J \subseteq \lambda$ ,  $\pi_J(a) \in \text{cl}_{\sigma_J}(\pi_J[A])$ . By lemma 4.4,  $t(\sigma_J) \leq \kappa$ , then we can take  $B_J \subseteq \pi_J[A]$  of cardinality  $\leq \kappa$  such that  $\pi_J(a) \in \text{cl}_{\sigma_J}(B_J)$ . For each  $b \in B_J$  choose  $x_b \in A$  such that  $\pi_J(x_b) = b$ , and let  $C_J = \{x_b : b \in B_J\}$ .

Now, let  $\mathcal{J}$  be a cofinal family in  $[\lambda]^{\aleph_0}$  and let

$$C = \bigcup\{C_J : J \in \mathcal{J}\}.$$

Note that  $|C| \leq \text{Cov}_\omega(\lambda) \cdot t(X) = \kappa$ . Then  $\text{cl}_{\sigma(\lambda)}(C) \subseteq A$ . So, it remains to be proved that  $a \in \text{cl}_{\sigma(\lambda)}(C)$ . Let  $U = \prod\{U_j : j \in J'\} \times \prod\{X_\alpha : \alpha \in \lambda \setminus J'\}$  be an basic neighborhood of  $a$  in  $\sigma(\lambda)$ . Let  $J \in \mathcal{J}$  such that  $J' \subseteq J$ . Since  $\pi_J(a) \in \text{cl}_{\sigma_J}(B_J) = \text{cl}_{\sigma_J}(\pi_J[C_J])$ , then  $\pi_J[U] \cap \pi_J[C_J] \neq \emptyset$ ; so  $U \cap C_J \neq \emptyset$ . Therefore,  $U \cap C \neq \emptyset$ .  $\square$

In the same way we have proved the lemma 4.4, we can show the following result for the cases in which  $\text{Cov}_\omega(t(X)) > t(X)$ :

**Lemma 4.10.** *If  $X$  is a Lindelöf  $P$ -space then*

$$t(\sigma(X^{\aleph_\omega}, x^*)_\delta) = t(X).$$

*Proof.* Let  $\kappa = t(X)$ . Suppose that  $\text{Cov}_\omega(\kappa) > \kappa$ . Note that  $\kappa \geq \aleph_\omega$ . Let  $Y$  be a non-closed subset of  $\sigma(X^\omega, x^*)_\delta$  and let  $q \in \text{cl}(Y) \setminus Y$ . For each  $n \in \omega$ , let

$$Y_n = \{y \in Y : \text{supp}(y) \subseteq \omega_n\}.$$

Since  $Y = \bigcup\{Y_n : n \in \omega\}$  and  $\sigma(X^\omega, x^*)_\delta$  is a  $P$ -space, there exists a  $m \in \omega$  such that  $q \in \text{cl}(Y_m)$ . Now,  $\pi_m(q) \in \text{cl}(\pi_m[Y_m])$ , where  $\pi_m$  is the natural projection from  $X^{\aleph_\omega}$  in  $X^{\aleph_m}$ . Since by the theorem 4.9 the tightness of  $X^{\aleph_m}$  is  $\kappa$ , there exists  $Z' \subseteq \pi_m[Y_m]$  of cardinality  $\leq \kappa$  such that  $\pi_m(q) \in \text{cl}(Z')$ . Then  $Z = Z' \times \prod\{\{x^*(n)\} : n \geq m\} \subseteq Y_m \subseteq Y$  and since  $\text{supp}(q) \subseteq \omega_m$ , we have  $q \in \text{cl}(Z)$ .  $\square$

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